

ANALYTIC STRUCTURE OF WEIGHTED SHIFTS ON DIRECTED TREES

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ABSTRACT. We show that a weighted shift on a directed tree is related to a multiplier algebra of coefficients of analytic functions. We use this relation to study spectral properties of the operators in question.

1. INTRODUCTION

Theory of analytic functions plays a central role in operator theory. It has been a source of methods, examples and problems, and has led to numerous important results. The study of the unilateral shift owes much of its success to use of the Hardy space methods (see the monograph [14]). The same applies to Toeplitz and Hankel operators or composition operators. Weighted shifts have also been studied with analytic function theory approach. An excellent exposition of an interplay between weighted shift operators and analytic functions has been given by A. Shields in [16]. Essential ingredients of the considerations therein were viewing a weighted shift operator as “multiplication by z ” on a Hilbert space consisting of formal power series and showing that the structure of this space is in fact analytic. This enabled using multiplication operators and bounded point evaluations, tools known to be very powerful in variety of problems in operator theory. Other notable examples of using analytic models of operators can be found in [5, 12, 22, 18, 13].

In the present paper we study a class of (bounded) weighted shifts on directed trees focusing on analytic aspects of their theory. The class generalizes classical weighted shifts (see [7]) and is a source of interesting examples (see e.g., [1, 2, 8, 9, 10, 11, 15, 21]). We start our investigations by introducing the notion of a weighted shift on a weighted directed tree (see Section 3). Although it is formally a generalization of a weighted shift on a (non-weighted) directed tree, both the notions are equivalent in a sense (see Proposition 28 and Remark 30). Nevertheless, the technical side of our considerations seems to be easier to handle in the “weighted directed tree” setting (in particular, this enables us to consider shifts with weights summing over children of every vertex up to 1, see condition $(\star\star)$, which is crucial to our study). Then, we define and study multiplier algebras related to weighted shifts on weighted directed trees (see Section 4). We show that these algebras consists of coefficients of analytic functions (see Propositions 6 and 7). Later, we introduce bounded point evaluations and study spectral properties of the adjoints of weighted shifts on weighted directed trees (see Section 5). The main result here provides a kind of functional calculus (see Theorem 19) for functions from multiplier algebras. Next, we investigate the point spectrum of the adjoint of a weighted shift on a weighted directed tree by looking at its behaviour on paths. We show that the spectrum contains a complex disc of radius which can be estimated by the supremum of appropriate limits along the paths (see Theorem 23). Moreover, we show that for particular classes of directed trees the closures of the spectrum and the disc are equal. We

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conclude the paper with results concerning weighted shifts on (non-weighted) directed trees (see Section 7).

2. PRELIMINARIES

Let \mathbb{N} , \mathbb{R} and \mathbb{C} denote the set of all natural numbers, real numbers and complex numbers, respectively. Set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{R}_+ = [0, \infty)$. For $\kappa \in \mathbb{N} \cup \{\infty\}$, J_κ stands for the set $\{n \in \mathbb{N}: n \leq \kappa\}$. If Y is a set, then $\text{card}(Y)$ denotes the cardinal number of Y . For any $r \in (0, \infty)$, Δ_r stands for $\{z \in \mathbb{C}: |z| < r\}$.

Let V be a nonempty set and $\beta = \{\beta_v\}_{v \in V}$ be a family of positive real numbers. Then $\ell^2(\beta) = \ell^2(V, \beta)$ denotes the Hilbert space of all functions $f: V \rightarrow \mathbb{C}$ such that $\sum_{v \in V} |f(v)|^2 \beta_v < \infty$ with the inner product given by $\langle f, g \rangle_\beta = \sum_{v \in V} f(v) \overline{g(v)} \beta_v$ for $f, g \in \ell^2(\beta)$. The norm induced by $\langle \cdot, \cdot \rangle_\beta$ is denoted by $\|\cdot\|_\beta$. For $u \in V$, we define $e_u \in \ell^2(\beta)$ to be the characteristic function of the one-point set $\{u\}$; clearly, $\{e_u\}_{u \in V}$ is an orthogonal basis of $\ell^2(\beta)$. We will denote by \mathcal{E} the linear span of $\{e_u\}_{u \in V}$. Given a subset V' of V , $\ell^2(V', \beta)$ stands for the subspace of $\ell^2(\beta)$ composed of all functions f such that $f(v) = 0$ for all $v \in V \setminus V'$, and $\mathcal{E}_{V'}$ denotes the set of all $f \in \ell^2(V', \beta)$ such that $\{v \in V: f(v) \neq 0\}$ is finite. By $\mathcal{Q}_{V'}$ we denote the orthogonal projection from $\ell^2(\beta)$ onto $\ell^2(V', \beta)$.

Let A be a (linear) operator in a (complex) Hilbert space \mathcal{H} . Then $\mathcal{D}(A)$, $r(A)$ and A^* denote the domain, the spectral radius and the adjoint of A , respectively (in case it exists). A linear subspace \mathcal{F} of $\mathcal{D}(A)$ is called a core of A if \mathcal{F} is dense in $\mathcal{D}(A)$ in the graph norm induced by A , i.e., the norm $\|\cdot\|_A$ given by $\|f\|_A^2 = \|Af\|^2 + \|f\|^2$, for $f \in \mathcal{D}(A)$. If \mathcal{F} is a subspace of \mathcal{H} , then $A|_{\mathcal{F}}$ is the operator in \mathcal{H} acting on the domain $\mathcal{D}(A|_{\mathcal{F}}) = \mathcal{F} \cap \mathcal{D}(A)$ according to the formula $A|_{\mathcal{F}} f = Af$. The algebra of all bounded operators on \mathcal{H} is denoted by $\mathbf{B}(\mathcal{H})$. If $A \in \mathbf{B}(\mathcal{H})$, then we denote by $\sigma(A)$ and $\sigma_p(A)$ the spectrum and point spectrum of A .

3. WEIGHTED SHIFTS ON DIRECTED TREES

Let $\mathcal{T} = (V, E)$ be a directed tree (V and E stand for the sets of vertices and directed edges of \mathcal{T} , respectively). Set $\text{Chi}(u) = \{v \in V: (u, v) \in E\}$ for $u \in V$. Denote by par the partial function from V to V which assigns to a vertex $u \in V$ its parent $\text{par}(u)$ (i.e. a unique $v \in V$ such that $(v, u) \in E$). For $k \in \mathbb{N}$, par^k denotes the k -fold composition of the partial function par ; par^0 denotes the identity map on V . A vertex $u \in V$ is called a *root* of \mathcal{T} if u has no parent. A root is unique (provided it exists); we denote it by root . The tree \mathcal{T} is *rooted* if the root exists. The tree \mathcal{T} is *leafless* if every vertex $v \in V$ has a child, i.e., $\text{card}(\text{Chi}(v)) \geq 1$. All the directed trees considered here are assumed to be rooted and countably infinite. In most of the cases they will be also leafless. We set $V^\circ = V \setminus \{\text{root}\}$. If $v \in V$, then $|v|$ denotes the unique $k \in \mathbb{N}_0$ such that $\text{par}^k(v) = \text{root}$. For given $u \in V$ and $n \in \mathbb{N}$ we denote by $\text{Des}^{(n)}(u)$ the set $\{v \in V: \text{par}^k(v) = u \text{ for some } k \in J_n \cup \{0\}\}$; in addition, we set $\text{Des}(u) = \{v \in V: \text{par}^k(v) = u \text{ for some } k \in \mathbb{N}_0\}$. In the paper we will also consider subgraphs of \mathcal{T} . On this occasion we will use the notions of children Chi , parent function par , the root root , etc., *with respect to a subgraph*, say \mathcal{G} ; they will be denoted by $\text{Chi}_{\mathcal{G}}$, $\text{par}_{\mathcal{G}}$, $\text{root}_{\mathcal{G}}$, etc.

Let $\mathcal{T} = (V, E)$ be a directed tree. A subgraph \mathcal{S} of \mathcal{T} which is a directed tree itself is called a *subtree* of \mathcal{T} . A *path* in \mathcal{T} is a subtree $\mathcal{P} = (V', E')$ of \mathcal{T} which satisfies the following two conditions: (i) $\text{root} \in \mathcal{P}$, (ii) for every $v \in V'$, $\text{card}(\text{Chi}_{\mathcal{P}}(v)) = 1$. The collection of all paths in \mathcal{T} is denoted by $\mathcal{P} = \mathcal{P}(\mathcal{T})$. Throughout the paper $\mathbb{1} = \mathbb{1}_V$ stands for either of the families

$\{\beta_v\}_{v \in V}$ or $\{\beta_v\}_{v \in V^\circ}$ with $\beta_v = 1$ for every $v \in V$. We refer the reader to [7] for more on directed trees.

Now, we give a definition of a weighted shift on a weighted directed tree. Formally the notion generalizes that of a weighted shift on a directed tree from [7] but in view of Proposition 28 and Remark 30 (see also Proposition 29) both the notions are equivalent. Let $\mathcal{T} = (V, E)$ be a directed tree and let $\lambda = \{\lambda_v\}_{v \in V^\circ} \subseteq \mathbb{C}$. We define then the map $A_\lambda^\lambda: \mathbb{C}^V \rightarrow \mathbb{C}^V$ via

$$(A_\lambda^\lambda f)(v) = \begin{cases} \lambda_v \cdot f(\text{par}(v)) & \text{if } v \in V^\circ, \\ 0 & \text{if } v = \text{root}. \end{cases}$$

Given $\beta = \{\beta_v\}_{v \in V} \subseteq \mathbb{C}$, we denote by \mathcal{T}_β the pair (\mathcal{T}, β) ; this is the *weighted tree* mentioned above. By a *weighted shift on \mathcal{T}_β with weights $\lambda = \{\lambda_v\}_{v \in V^\circ} \subseteq \mathbb{C}$* , we mean the operator S_λ in $\ell^2(\beta)$ defined as follows

$$\begin{aligned} \mathcal{D}(S_\lambda) &= \{f \in \ell^2(\beta): A_\lambda^\lambda f \in \ell^2(\beta)\}, \\ S_\lambda f &= A_\lambda^\lambda f, \quad f \in \mathcal{D}(S_\lambda). \end{aligned}$$

Note that the weighted shift S_λ on \mathcal{T}_β (in a sense of the above definition) is the *weighted shift S_λ on the directed tree \mathcal{T}* as in [7].

The questions of boundedness of a k -th power of S_λ on \mathcal{T}_β and evaluation of its norm can be answered in terms of the products of k consecutive weights. To state the result we need some notation. Let $\mathcal{T} = (V, E)$ be a directed tree and let $\lambda = \{\lambda_v\}_{v \in V^\circ} \subseteq \mathbb{C}$. For $u \in V$ and $v \in \text{Des}(u)$ we define

$$\lambda_{u|v} = \lambda_{u|v}^\mathcal{T} = \begin{cases} 1 & \text{if } u = v, \\ \prod_{j=0}^{n-1} \lambda_{\text{par}^j(v)} & \text{if } v \in \text{Chi}^{(n)}(u). \end{cases}$$

Arguing as in the proof of [7, Lemmata 3.1.8 and 6.1.1], where $k = 1$ and $\beta = \mathbf{1}$ were considered, we get the following.

Lemma 1. *Let $\mathcal{T} = (V, E)$ be a directed tree. Let $\lambda = \{\lambda_v\}_{v \in V^\circ} \subseteq \mathbb{C}$ and $\beta = \{\beta_v\}_{v \in V} \subseteq (0, \infty)$. Then the weighted shift S_λ on \mathcal{T}_β is bounded on $\ell^2(\beta)$ if and only if*

$$\sup_{u \in V} \sum_{v \in \text{Chi}(u)} |\lambda_v|^2 \frac{\beta_v}{\beta_u} < \infty.$$

Moreover, if S_λ is bounded, then

$$\|S_\lambda^k\|^2 = \sup_{u \in V} \sum_{v \in \text{Chi}^{(k)}(u)} |\lambda_{u|v}|^2 \frac{\beta_v}{\beta_u}, \quad k \in \mathbb{N}.$$

In the course of our study we will use the formula for the adjoint S_λ^* of the weighted shift S_λ on \mathcal{T}_β , which is given below. For our purposes it suffices to have it in the bounded case. It can be proofed as in [7, Proposition 3.4.1(ii)].

Lemma 2. *Let $\mathcal{T} = (V, E)$ be a directed tree, $\lambda = \{\lambda_v\}_{v \in V^\circ} \subseteq \mathbb{C}$ and let $\beta = \{\beta_v\}_{v \in V} \subseteq (0, \infty)$. If S_λ is the weighted shift on \mathcal{T}_β such that $S_\lambda \in \mathbf{B}(\ell^2(\beta))$, then its adjoint S_λ^* is given by*

$$S_\lambda^* e_u = \begin{cases} \lambda_u \frac{\beta_u}{\beta_{\text{par}(u)}} e_{\text{par}(u)} & \text{if } u \in V^\circ, \\ 0 & \text{if } u = \text{root}. \end{cases}$$

4. MULTIPLICATION OPERATORS

Multiplication operators on weighted sequence spaces are helpful when studying “classical” weighted shifts (see [16]). As it turns out, they can be used also in the context of weighted shifts on directed trees. In this section we define counterparts of classical multiplication operators related to weighted shifts on directed trees.

We note that some of the results here are valid in the more general context of reproducing kernel Hilbert spaces (see Acknowledgments) but, for the sake of consistency with subsequent results, we opt for a direct approach and present them in a specific form (for many interesting facts concerning multiplication operators and multipliers in RKHS we refer the reader to [19, 20]).

It will be assumed throughout the section¹ (if not stated differently).

- (\star) $\mathcal{T} = (V, E)$ is a countably infinite rooted and leafless directed tree,
 $\beta = \{\beta_v\}_{v \in V} \subseteq (0, \infty)$ and $\{\lambda_v\}_{v \in V^\circ} \subseteq (0, \infty)$.

We start with defining a multiplication operator. Let $\hat{\varphi}: \mathbb{N}_0 \rightarrow \mathbb{C}$. Define $\Gamma_{\hat{\varphi}}^\lambda: \mathbb{C}^V \rightarrow \mathbb{C}^V$ by

$$(\Gamma_{\hat{\varphi}}^\lambda f)(v) = \sum_{k=0}^{|v|} \lambda_{\text{par}^k(v)|v} \hat{\varphi}(k) f(\text{par}^k(v)), \quad v \in V.$$

The multiplication operator $M_{\hat{\varphi}}^{\lambda, \beta}: \ell^2(\beta) \supseteq \mathcal{D}(M_{\hat{\varphi}}^{\lambda, \beta}) \rightarrow \ell^2(\beta)$ is given by

$$\begin{aligned} \mathcal{D}(M_{\hat{\varphi}}^{\lambda, \beta}) &= \{f \in \ell^2(\beta): \Gamma_{\hat{\varphi}}^\lambda f \in \ell^2(\beta)\}, \\ M_{\hat{\varphi}}^{\lambda, \beta} f &= \Gamma_{\hat{\varphi}}^\lambda f, \quad f \in \mathcal{D}(M_{\hat{\varphi}}^{\lambda, \beta}). \end{aligned}$$

The function $\hat{\varphi}: \mathbb{N}_0 \rightarrow \mathbb{C}$ is said to be the *symbol* of $M_{\hat{\varphi}}^{\lambda, \beta}$. If no confusion can arise, we write $\Gamma_{\hat{\varphi}}$ and $M_{\hat{\varphi}}$ instead of $\Gamma_{\hat{\varphi}}^\lambda$ and $M_{\hat{\varphi}}^{\lambda, \beta}$, respectively.

As shown in Lemma 3 below, any multiplication operator $M_{\hat{\varphi}}$ is automatically closed.

Lemma 3. *Assume (\star). For every $\hat{\varphi}: \mathbb{N}_0 \rightarrow \mathbb{C}$, the multiplication operator $M_{\hat{\varphi}}$ is closed.*

Proof. Let $\{f_n\}_{n=1}^\infty \subseteq \mathcal{D}(M_{\hat{\varphi}})$ and $f, g \in \ell^2(\beta)$ satisfy $\lim_{n \rightarrow \infty} f_n = f$ and $\lim_{n \rightarrow \infty} M_{\hat{\varphi}} f_n = g$. Then for every $v \in V$, $\lim_{n \rightarrow \infty} f_n(v) = f(v)$, which implies that

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{|v|} \lambda_{\text{par}^k(v)|v} \hat{\varphi}(k) f_n(\text{par}^k(v)) = \sum_{k=0}^{|v|} \lambda_{\text{par}^k(v)|v} \hat{\varphi}(k) f(\text{par}^k(v)), \quad v \in V.$$

On the other hand, we have

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{|v|} \lambda_{\text{par}^k(v)|v} \hat{\varphi}(k) f_n(\text{par}^k(v)) = \lim_{n \rightarrow \infty} (M_{\hat{\varphi}} f_n)(v) = g(v), \quad v \in V.$$

Hence, $\Gamma_{\hat{\varphi}} f = g$. Thus $f \in \mathcal{D}(M_{\hat{\varphi}})$ and $M_{\hat{\varphi}} f = g$, which completes the proof. \square

Let $\mathcal{M}(\mathcal{T}_\beta, \lambda)$ denote the collection of all functions $\hat{\varphi}: \mathbb{N}_0 \rightarrow \mathbb{C}$ such that $\mathcal{D}(M_{\hat{\varphi}}) = \ell^2(\beta)$. Clearly, $\mathcal{M}(\mathcal{T}_\beta, \lambda)$ is a linear space (with standard addition of functions).

Lemma 4. *Assume (\star). For every $\hat{\varphi} \in \mathcal{M}(\mathcal{T}_\beta, \lambda)$, the multiplication operator $M_{\hat{\varphi}}$ is bounded on $\ell^2(\beta)$.*

Proof. It is well-known that every closed everywhere defined Banach space operator is automatically bounded ([3, Theorem III.12.6]). Thus, we deduce boundedness of $M_{\hat{\varphi}}$ from Lemma 3. \square

¹In view of Propositions 28 and 31 the assumption $\{\lambda_v\}_{v \in V^\circ} \subseteq (0, \infty)$ is not restrictive (cf. Remark 33).

In view of the above, $\mathcal{M}(\mathcal{T}_\beta, \lambda)$ is a normed space with the norm $\|\cdot\|$ given by

$$\|\hat{\varphi}\| \stackrel{df}{=} \|M_{\hat{\varphi}}\|, \quad \hat{\varphi} \in \mathcal{M}(\mathcal{T}_\beta, \lambda).$$

The space $\mathcal{M}(\mathcal{T}_\beta, \lambda)$ turns out to have a natural Banach algebra structure. It suffices to endow it with the Cauchy-type multiplication $*$: $\mathbb{C}^{\mathbb{N}_0} \times \mathbb{C}^{\mathbb{N}_0} \rightarrow \mathbb{C}^{\mathbb{N}_0}$ given by

$$(1) \quad (\hat{\varphi} * \hat{\psi})(k) = \sum_{j=0}^k \hat{\varphi}(j) \hat{\psi}(k-j), \quad \hat{\varphi}, \hat{\psi} \in \mathbb{C}^{\mathbb{N}_0}.$$

Theorem 5. *Let $\mathcal{T} = (V, E)$ be a countably infinite rooted and leafless directed tree, $\beta = \{\beta_v\}_{v \in V} \subseteq (0, \infty)$ and $\{\lambda_v\}_{v \in V^\circ} \subseteq (0, \infty)$. Let $S_\lambda \in \mathbf{B}(\ell^2(\beta))$ be a weighted shift on \mathcal{T}_β . Then following assertions are satisfied:*

- (i) *For every $n \in \mathbb{N}_0$, $M_{\chi_{\{n\}}} = S_\lambda^n$.*
- (ii) *If $\hat{\varphi}: \mathbb{N}_0 \rightarrow \mathbb{C}$ has finite support, then $\hat{\varphi} \in \mathcal{M}(\mathcal{T}_\beta, \lambda)$.*
- (iii) *For all $\hat{\varphi}, \hat{\psi} \in \mathcal{M}(\mathcal{T}_\beta, \lambda)$, the function $\hat{\varphi} * \hat{\psi}$ belongs to $\mathcal{M}(\mathcal{T}_\beta, \lambda)$ and*

$$M_{\hat{\varphi}} M_{\hat{\psi}} = M_{\hat{\varphi} * \hat{\psi}}.$$

- (iv) *The space $\mathcal{M}(\mathcal{T}_\beta, \lambda)$, endowed with the Cauchy-type multiplication, is a commutative Banach algebra with unit.*

Proof. (i) This follows directly from the definitions of $M_{\hat{\varphi}}$ and S_λ .

(ii) This follows from (i) and linearity.

(iii) We first show that

$$(2) \quad \Gamma_{\hat{\varphi}} \Gamma_{\hat{\psi}} f = \Gamma_{\hat{\varphi} * \hat{\psi}} f, \quad f \in \mathbb{C}^V \text{ and } \hat{\varphi}, \hat{\psi} \in \mathbb{C}^{\mathbb{N}_0}.$$

Fix $\hat{\varphi}, \hat{\psi} \in \mathbb{C}^V$ and take $f \in \mathbb{C}^V$. Note that for every $v \in V$ and all $k \in \{0, \dots, |v|\}$ and $j \in \{0, \dots, |\text{par}^k(v)|\}$, the expression $\text{par}^j(\text{par}^k(v))$ makes sense and equals $\text{par}^{j+k}(v)$; moreover $|\text{par}^k(v)| = |v| - k$. Hence, we have the equality

$$\begin{aligned} (\Gamma_{\hat{\varphi}} \Gamma_{\hat{\psi}} f)(v) &= \sum_{k=0}^{|v|} \lambda_{\text{par}^k(v)|v} \hat{\varphi}(k) (\Gamma_{\hat{\psi}} f)(\text{par}^k(v)) \\ &= \sum_{k=0}^{|v|} \lambda_{\text{par}^k(v)|v} \hat{\varphi}(k) \sum_{j=0}^{|\text{par}^k(v)|} \lambda_{\text{par}^j(\text{par}^k(v))|\text{par}^k(v)} \hat{\psi}(j) f(\text{par}^j \text{par}^k(v)) \\ &= \sum_{k=0}^{|v|} \lambda_{\text{par}^k(v)|v} \hat{\varphi}(k) \sum_{j=0}^{|v|-k} \lambda_{\text{par}^{j+k}(v)|\text{par}^k(v)} \hat{\psi}(j) f(\text{par}^{j+k}(v)) \\ &= \sum_{k=0}^{|v|} \sum_{j=k}^{|v|} \lambda_{\text{par}^k(v)|v} \lambda_{\text{par}^j(v)|\text{par}^k(v)} \hat{\varphi}(k) \hat{\psi}(j-k) f(\text{par}^j(v)), \\ (3) \quad &= \sum_{k=0}^{|v|} \sum_{j=k}^{|v|} \lambda_{\text{par}^j(v)|v} \hat{\varphi}(k) \hat{\psi}(j-k) f(\text{par}^j(v)), \quad v \in V. \end{aligned}$$

On the other hand, we see that

$$(4) \quad (\Gamma_{\hat{\varphi} * \hat{\psi}} f)(v) = \sum_{j=0}^{|v|} \lambda_{\text{par}^j(v)|v} \sum_{k=0}^j \hat{\varphi}(k) \hat{\psi}(j-k) f(\text{par}^j(v)), \quad v \in V.$$

By (3) and (4), we get

$$(\Gamma_{\hat{\varphi}} \Gamma_{\hat{\psi}} f)(v) = (\Gamma_{\hat{\varphi} * \hat{\psi}} f)(v), \quad v \in V.$$

Hence $\mathcal{D}(M_{\hat{\varphi} * \hat{\psi}}) = \ell^2(\beta)$ and $\mathcal{D}(M_{\hat{\varphi} * \hat{\psi}}) = M_{\hat{\varphi}} M_{\hat{\psi}}$, which proves (iii).

(iv) In view of (1), (i) and (iii), $\mathcal{M}(\mathcal{T}_\beta, \lambda)$ is a commutative algebra with unit. We prove now that $\mathcal{M}(\mathcal{T}_\beta, \lambda)$ is closed. To this end take a Cauchy sequence $\{\hat{\varphi}_n\}_{n=1}^\infty \subseteq \mathcal{M}(\mathcal{T}_\beta, \lambda)$. Then $\{M_{\hat{\varphi}_n}\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbf{B}(\ell^2(\beta))$, and hence there is $A \in \mathbf{B}(\ell^2(\beta))$ such that $\lim_{n \rightarrow \infty} M_{\hat{\varphi}_n} = A$. We note that

$$(5) \quad \frac{(Ae_{\text{root}})(u_1)}{\lambda_{\text{root}|u_1}} = \frac{(Ae_{\text{root}})(u_2)}{\lambda_{\text{root}|u_2}}, \quad \text{if } u_1, u_2 \in V \text{ satisfy } |u_1| = |u_2|,$$

which easily follows from

$$(6) \quad (Ae_{\text{root}})(v) = \lim_{n \rightarrow \infty} \sum_{k=0}^{|v|} \lambda_{\text{par}^k(v)|v} \hat{\varphi}_n(k) e_{\text{root}}(\text{par}^k(v)) = \lim_{n \rightarrow \infty} \lambda_{\text{root}|v} \hat{\varphi}_n(|v|), \quad v \in V.$$

Let us define the function $\hat{\psi}: \mathbb{N}_0 \rightarrow \mathbb{C}$ by

$$(7) \quad \hat{\psi}(|v|) = \frac{(Ae_{\text{root}})(v)}{\lambda_{\text{root}|v}}.$$

By (5), $\hat{\psi}$ is well-defined. Moreover, by (6) and (7), we have

$$\begin{aligned} (Af)(v) &= \lim_{n \rightarrow \infty} (M_{\hat{\varphi}_n} f)(v) = \lim_{n \rightarrow \infty} \sum_{k=0}^{|v|} \lambda_{\text{par}^k(v)|v} \hat{\varphi}_n(k) f(\text{par}^k(v)) \\ &= \sum_{k=0}^{|v|} \lambda_{\text{par}^k(v)|v} \hat{\psi}(k) f(\text{par}^k(v)) = (\Gamma_{\hat{\psi}} f)(v), \quad v \in V. \end{aligned}$$

Since A is bounded, $\mathcal{D}(M_{\hat{\psi}}) = \ell^2(\beta)$ and $A = M_{\hat{\psi}}$. Thus $\hat{\psi} \in \mathcal{M}(\mathcal{T}_\beta, \lambda)$. Since, $\|\hat{\psi} - \hat{\varphi}_n\| = \|M_{\hat{\psi}} - M_{\hat{\varphi}_n}\| \rightarrow 0$ as $n \rightarrow \infty$, we get the claim. \square

The above justifies calling $\mathcal{M}(\mathcal{T}_\beta, \lambda)$ the *multiplier algebra* induced by S_λ .

Proposition 6. *Assume (\star) . Let $S_\lambda \in \mathbf{B}(\ell^2(\beta))$ be a weighted shift on \mathcal{T}_β . Then for every $\hat{\varphi} \in \mathcal{M}(\mathcal{T}_\beta, \lambda)$ and every $w \in \Delta_{r(S_\lambda)}$ the series $\sum_{n \in \mathbb{N}_0} \hat{\varphi}(n) w^n$ is absolutely convergent.*

Proof. For $u \in V$ and $k \in \mathbb{N}_0$ let $f_{u,k}: V \rightarrow \mathbb{C}$ be a function defined by

$$f_{u,k}(v) = \begin{cases} \lambda_{u|v} & \text{if } v \in \text{Chi}^{(k)}(u), \\ 0 & \text{otherwise.} \end{cases}$$

Let us notice that by Lemma 1 we have

$$\sum_{v \in V} (f_{u,k}(v))^2 \beta_v = \sum_{v \in \text{Chi}^{(k)}(u)} \lambda_{u|v}^2 \beta_v \leq \beta_u \|S_\lambda^k\|^2,$$

which implies that $f_{u,k} \in \ell^2(\beta)$ and $\|f_{u,k}\| \leq \sqrt{\beta_u} \|S_\lambda^k\|$. Moreover, we have

$$\begin{aligned} |\hat{\varphi}(k) \sum_{v \in \text{Chi}^{(k)}(u)} \lambda_{u|v}^2 \beta_v| &= \left| \sum_{v \in \text{Chi}^{(k)}(u)} \lambda_{u|v} \hat{\varphi}(k) f_{u,k}(v) \beta_v \right| \\ &= |\langle M_{\hat{\varphi}} e_u, f_{u,k} \rangle_\beta| \leq \sqrt{\beta_u} \|M_{\hat{\varphi}}\| \|f_{u,k}\|_\beta \leq \beta_u \|M_{\hat{\varphi}}\| \|S_\lambda^k\|. \end{aligned}$$

Thus, taking supremum over $u \in V$ and using Lemma 1, we get $\|S_\lambda^k\|^2 |\hat{\varphi}(k)| \leq \|M_{\hat{\varphi}}\| \|S_\lambda^k\|$. Consequently, $|\hat{\varphi}(k)| \leq \|M_{\hat{\varphi}}\| \|S_\lambda^k\|^{-1}$ for every $k \in \mathbb{N}_0$. This together with Gelfand's formula for the spectral radius gives the claim. \square

Proposition 7. *Assume (\star) . Let $S_\lambda \in \mathbf{B}(\ell^2(\beta))$ be a weighted shift on \mathcal{T}_β . Let $\hat{\varphi}: \mathbb{N}_0 \rightarrow \mathbb{C}$ be such that the series $\sum_{k=0}^\infty \hat{\varphi}(k) z^n$ is convergent for every $z \in \Delta_{\|S_\lambda\|}$. If the function $\varphi: \Delta_{\|S_\lambda\|} \rightarrow \mathbb{C}$ given by $\varphi(z) = \sum_{k=0}^\infty \hat{\varphi}(k) z^n$ is bounded, then $\hat{\varphi} \in \mathcal{M}(\mathcal{T}_\beta, \lambda)$ and $\|M_{\hat{\varphi}}\| \leq \|\varphi\|_\infty := \sup\{|\varphi(z)| : |z| < \|S_\lambda\|\}$.*

Proof. For $k \in \mathbb{N}$ we define functions $\hat{\omega}_k: \mathbb{N}_0 \rightarrow \mathbb{C}$ and $\omega_k: \Delta_{\|S_\lambda\|} \rightarrow \mathbb{C}$ by

$$\hat{\omega}_k(n) = \begin{cases} \frac{k+1-n}{k+1} \hat{\varphi}(n) & \text{if } n \leq k+1, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\omega_k(z) = \sum_{n=0}^\infty \hat{\omega}_k(n) z^n,$$

It is easily seen that

$$\omega_k(z) = \frac{1}{k+1} \sum_{n=0}^k s_n(z), \quad z \in \Delta_{\|S_\lambda\|},$$

where $s_n: \Delta_{\|S_\lambda\|} \rightarrow \mathbb{C}$, $n \in \mathbb{N}_0$, are given by $s_n(z) = \sum_{k=0}^n \hat{\varphi}(k) z^k$. Hence by Theorem 5, the von Neumann inequality and the well-known facts about Cesàro means (see [6, p. 16-24]) we have

$$\sup_{k \in \mathbb{N}_0} \|M_{\hat{\omega}_k}\| = \sup_{k \in \mathbb{N}_0} \|\omega_k(S_\lambda)\| \leq \sup\{\omega_k(z) : z \in \Delta_{\|S_\lambda\|}, k \in \mathbb{N}_0\} \leq \|\varphi\|_\infty.$$

Moreover, for every $f \in \ell^2(\beta)$ and every finite $W \subseteq V$ we have

$$(8) \quad \sum_{v \in W} \left| (M_{\hat{\omega}_k} f)(v) \right|_{\beta_v}^2 \leq \sup_{n \in \mathbb{N}_0} \|M_{\hat{\omega}_n}\|^2 \|f\|_\beta^2 \leq \|\varphi\|_\infty^2 \|f\|_\beta^2, \quad k \in \mathbb{N}_0.$$

Since for every $v \in V$, $\lim_{k \rightarrow \infty} (M_{\hat{\omega}_k} f)(v) = (\Gamma_{\hat{\varphi}} f)(v)$, we infer from (8) that

$$\sum_{v \in V} \left| (\Gamma_{\hat{\varphi}} f)(v) \right|_{\beta_v}^2 \leq \|\varphi\|_\infty^2 \|f\|_\beta^2.$$

Hence, $\mathcal{D}(M_{\hat{\varphi}}) = \ell^2(\beta)$ and thus $\hat{\varphi} \in \mathcal{M}(\mathcal{T}_\beta, \lambda)$. \square

In view of Proposition 7, if $r(S_\lambda) = \|S_\lambda\|$ and $\hat{\varphi}: \mathbb{N}_0 \rightarrow \mathbb{C}$ induces the function φ which is analytic in Δ_r with $r > r(S_\lambda)$, then $\hat{\varphi} \in \mathcal{M}(\mathcal{T}_\beta, \lambda)$. On the other hand, if $r(S_\lambda) < \|S_\lambda\|$, then one can find an analytic function ψ on Δ_r , with $r(S_\lambda) < r < \|S_\lambda\|$, without analytic extension onto $\Delta_{\|S_\lambda\|}$. Proposition 7 cannot be used to verify if the function $\hat{\psi}$ determined by the coefficients of the series expansion of ψ at 0 belongs to $\mathcal{M}(\mathcal{T}_\beta, \lambda)$. One can use the following then (this fact can be proved with help of the Riesz functional calculus but we use approach based on the Gelfand formula for the spectral radius).

Proposition 8. *Assume (\star) . Let $S_\lambda \in \mathbf{B}(\ell^2(\beta))$ be a weighted shift on \mathcal{T}_β and let $r \in (r(S_\lambda), \infty)$. If $\hat{\varphi}: \mathbb{N}_0 \rightarrow \mathbb{C}$ is such that the series $\sum_{k=0}^\infty \hat{\varphi}(k) z^n$ is convergent for every $z \in \Delta_r$, then $\hat{\varphi} \in \mathcal{M}(\mathcal{T}_\beta, \lambda)$, the series $\sum_{k=0}^\infty \hat{\varphi}(k) S_\lambda^k$ is norm convergent and $M_{\hat{\varphi}} = \sum_{k=0}^\infty \hat{\varphi}(k) S_\lambda^k$.*

Proof. Since $r > r(S_\lambda)$, by the Gelfand formula for the spectral radius there exists $\rho \in (0, 1)$ and $k_0 \in \mathbb{N}$ such that

$$\frac{\rho}{\sqrt[k]{|\hat{\varphi}(k)|}} \geq \sqrt[k]{\|S_\lambda^k\|}, \quad k \geq k_0,$$

which implies that

$$|\hat{\varphi}(k)| \|S_\lambda^k\| \leq \rho^k, \quad k \geq k_0.$$

Hence, the series $\sum_{k=0}^\infty \hat{\varphi}(k) S_\lambda^k$ is norm convergent to some $S \in \mathbf{B}(\ell^2(\beta))$. Since $\mathcal{M}(\mathcal{T}_\beta, \lambda)$ is a Banach algebra, $S = M_{\hat{\psi}}$ with some $\hat{\psi}: \mathbb{N}_0 \rightarrow \mathbb{C}$. Now, evaluating $\langle M_{\hat{\psi}} e_{\text{root}}, e_u \rangle$, $u \in V$, we deduce that $\hat{\psi}(k) = \hat{\varphi}(k)$ for every $k \in \mathbb{N}_0$ and consequently $M_{\hat{\varphi}} = M_{\hat{\psi}} = \sum_{k=0}^\infty \hat{\varphi}(k) S_\lambda^k$. \square

In view of the above the following problem seems to be interesting.

Problem 9. Let $\hat{\varphi}: \mathbb{N}_0 \rightarrow \mathbb{C}$ be such that the series $\sum_{k=0}^\infty \hat{\varphi}(k) z^k$ is convergent for every $z \in \Delta_{r(S_\lambda)}$ and the function $\varphi: \Delta_{r(S_\lambda)} \rightarrow \mathbb{C}$ given by $\varphi(z) = \sum_{k=0}^\infty \hat{\varphi}(k) z^k$ is bounded. Does this imply that $\hat{\varphi} \in \mathcal{M}(\mathcal{T}_\beta, \lambda)$?

We finish the section with an auxiliary result, in view of which a multiplication operator with positive symbol can be effectively approximated with multiplication operators with finitely supported symbols.

Proposition 10. Assume (\star) . Let $\hat{\varphi} = |\hat{\varphi}| \in \mathcal{M}(\mathcal{T}_\beta, \lambda)$ and let $\hat{\varphi}^{(n)}: \mathbb{N}_0 \rightarrow \mathbb{C}$, $n \in \mathbb{N}$, be given by $\hat{\varphi}^{(n)}(k) = \chi_{\{1, \dots, n\}}(k) \hat{\varphi}(k)$, $k \in \mathbb{N}$. Then $\hat{\varphi}^{(m)} \in \mathcal{M}(\mathcal{T}_\beta, \lambda)$ for every $m \in \mathbb{N}$, and $\{M_{\hat{\varphi}^{(n)}}\}_{n=1}^\infty$ converges to $M_{\hat{\varphi}}$ in the strong operator topology.

Proof. The first part of the claim follows from Theorem 5. Now, let $f \in \ell^2(\beta)$. For every $v \in V$ we have

$$(M_{\hat{\varphi}^{(n)}} f)(v) = \sum_{k=0}^{|v|} \lambda_{\text{par}^k(v)|v} \hat{\varphi}^{(n)}(k) f(\text{par}^k(v)) \xrightarrow{n \rightarrow \infty} \sum_{k=0}^{|v|} \lambda_{\text{par}^k(v)|v} \hat{\varphi}(k) f(\text{par}^k(v)) = (M_{\hat{\varphi}} f)(v).$$

Since for every $f \in \ell^2(\beta)$, $M_{\hat{\varphi}}|f| \in \ell^2(\beta)$ and

$$\sum_{v \in V} |(M_{\hat{\varphi}^{(n)}} f)(v)|^2 \beta_v \leq \sum_{v \in V} \left(\sum_{k=0}^{|v|} \lambda_{\text{par}^k(v)|v} \hat{\varphi}(k) |f(\text{par}^k(v))| \right)^2 \beta_v = \|M_{\hat{\varphi}}|f|\|^2,$$

the claim follows from the Lebesgue dominated convergence theorem. \square

Problem 11. Does there exist $\hat{\varphi} \in \mathcal{M}(\mathcal{T}_\beta, \lambda)$ such that $|\hat{\varphi}| \notin \mathcal{M}(\mathcal{T}_\beta, \lambda)$?

5. BOUNDED POINT EVALUATIONS

In this section we define and study bounded point evaluations related to weighted shifts on weighted directed trees. In the classical weighted shift setting they were used successfully in many problems, for example to show that weighted shifts are reflexive (see [16]).

Theorem 12. Suppose $\mathcal{T} = (V, E)$ is a countably infinite rooted directed tree and $\beta = \{\beta_v\}_{v \in V} \subseteq (0, \infty)$. Let $w \in \mathbb{C}$. Then the following conditions are equivalent:

(i) There exists a continuous linear mapping $V_w: \ell^2(\beta) \rightarrow \mathbb{C}$ such that

$$(9) \quad V_w(f) = \sum_{v \in V} f(v) w^{|v|}, \quad f \in \mathcal{E}.$$

(ii) There exists $c > 0$ such that

$$\left| \sum_{v \in V} f(v) w^{|v|} \right| \leq c \|f\|_{\beta}, \quad f \in \mathcal{E}.$$

(iii) There exists $k_w \in \ell^2(\beta)$ such that

$$(10) \quad \langle f, k_w \rangle_{\beta} = \sum_{v \in V} f(v) w^{|v|}, \quad f \in \mathcal{E}.$$

(iv) $\sum_{v \in V} \beta_v^{-1} |w|^{2|v|} < \infty$.

Moreover, if any of the conditions (i)-(iv) holds, then both the mapping V_w and the function k_w are unique,

$$(11) \quad k_w(v) = \frac{\overline{w}^{|v|}}{\beta_v}, \quad v \in V,$$

(with the convention $0^0 = 1$) and

$$(12) \quad V_w(f) = \langle f, k_w \rangle_{\beta} = \sum_{v \in V} f(v) w^{|v|}, \quad f \in \ell^2(\beta),$$

with the series $\sum_{v \in V} f(v) w^{|v|}$ being absolutely summable.

Proof. If (iv) holds, then the function $k_w: V \rightarrow \mathbb{C}$ defined by (11) belongs to $\ell^2(\beta)$. Clearly, for every $u \in V$ it satisfies $\langle e_u, k_w \rangle_{\beta} = w^{|u|}$, which, by linearity, gives (10) and proves (iii). The uniqueness of k_w follows easily from (10).

That (iii) implies (ii) follows immediately from the Cauchy-Schwartz inequality.

If (ii) is satisfied, then the density of \mathcal{E} in $\ell^2(\beta)$ implies that for every $f \in \ell^2(\beta)$ the series $\sum_{v \in V} f(v) w^{|v|}$ is absolutely summable and $\sum_{v \in V} |f(v)| |w|^{|v|} \leq c \|f\|_{\beta}$. Thus, the mapping

$$\mathcal{E} \ni f \mapsto \sum_{v \in V} f(v) w^{|v|} \in \mathbb{C}$$

can be extended (in a unique way) to a continuous linear mapping $V_w: \ell^2(\beta) \rightarrow \mathbb{C}$, which gives (i).

If (i) is satisfied, then by the Riesz theorem there exists $k \in \ell^2(\beta)$ such that $V_w(f) = \langle f, k \rangle_{\beta}$ for all $f \in \ell^2(\beta)$. This and (9) yield $k = k_w$, with k_w given by (11). Hence $k_w \in \ell^2(\beta)$, which is equivalent to (iv). This completes the proof. \square

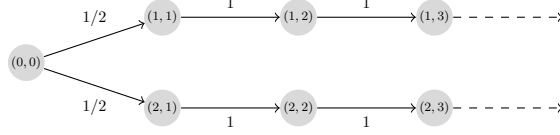
Under the assumption of Theorem 12, if $w \in \mathbb{C}$ and any of the conditions (i)-(iv) is satisfied, then we call w a *bounded point evaluation on \mathcal{T}_{β}* . By $\text{bpe}(\mathcal{T}_{\beta})$ we denote the set of all bounded point evaluations on \mathcal{T}_{β} .

Corollary 13. Suppose $\mathcal{T} = (V, E)$ is a countably infinite rooted directed tree and $\beta = \{\beta_v\}_{v \in V} \subseteq (0, \infty)$. Then the set $\text{bpe}(\mathcal{T}_{\beta})$ is circular.

Throughout the rest of this section we will assume additionally to (\star) that S_{λ} is bounded on $\ell^2(\beta)$ and the weights λ are distributed on \mathcal{T} in a special way. These assumptions are gathered in the following.

$$(\star\star) \quad \begin{aligned} \mathcal{T} &= (V, E) \text{ is a countably infinite rooted and leafless directed tree,} \\ \beta &= \{\beta_v\}_{v \in V} \subseteq (0, \infty) \text{ and } \lambda = \{\lambda_v\}_{v \in V^{\circ}} \subseteq (0, \infty) \text{ satisfy} \\ &\sup_{u \in V} \sum_{v \in \text{Chi}(u)} |\lambda_v|^2 \frac{\beta_v}{\beta_u} < \infty \text{ and } \sum_{v \in \text{Chi}(u)} \lambda_v = 1 \text{ for every } u \in V. \end{aligned}$$

Proposition 14. Assume $(\star\star)$. Let S_{λ} be a weighted shift on \mathcal{T}_{β} . If $w \in \mathbb{C}$ satisfies $|w| = \|S_{\lambda}\|$, then $w \notin \text{bpe}(\mathcal{T}_{\beta})$.

FIGURE 1. Tree $\mathcal{T}_{2,0}$

Proof. Let $|w| = \|S_\lambda\|$. By Lemma 1, for every $u \in V$, $|w|^2 \geq \sum_{v \in \text{Chi}(u)} \lambda_v^2 \frac{\beta_v}{\beta_u}$. Hence, by the Cauchy-Schwarz inequality we have

$$1 = \sum_{v \in \text{Chi}(u)} \lambda_v \leq \left(\sum_{v \in \text{Chi}(u)} \lambda_v^2 \frac{\beta_v}{\beta_u} \right) \left(\sum_{v \in \text{Chi}(u)} \frac{\beta_u}{\beta_v} \right) \leq \sum_{v \in \text{Chi}(u)} \frac{|w|^2}{\beta_v} \beta_u, \quad u \in V.$$

This in turn implies that

$$\sum_{v \in \text{Chi}(u)} \frac{|w|^{2|u|+2}}{\beta_v} = \frac{|w|^{2|u|}}{\beta_u} \sum_{v \in \text{Chi}(u)} \frac{|w|^2}{\beta_v} \beta_u \geq \frac{|w|^{2|u|}}{\beta_u}, \quad u \in V.$$

Thus

$$\sum_{|v|=k+1} \frac{|w|^{2k+2}}{\beta_v} \geq \sum_{|v|=k} \frac{|w|^{2k}}{\beta_v}, \quad k \in \mathbb{N},$$

which according to Theorem 12 (iv) proves that $w \notin \text{bpe}(\mathcal{T}_\beta)$. \square

Lemma 15. Assume $(\star\star)$. Let S_λ be a weighted shift on \mathcal{T}_β . If $w \in \text{bpe}(\mathcal{T}_\beta)$, then

$$(13) \quad \mathbf{V}_w(S_\lambda f) = w \mathbf{V}_w(f), \quad f \in \ell^2(\beta).$$

Proof. Since $\sum_{s \in \text{Chi}(u)} \lambda_s = 1$ for every $u \in V$, by Theorem 12, we have

$$\begin{aligned} \mathbf{V}_w(S_\lambda e_u) &= \sum_{v \in V} (S_\lambda e_u)(v) w^{|v|} = \sum_{v \in V} \left(\sum_{s \in \text{Chi}(u)} \lambda_s e_s \right)(v) w^{|v|} \\ &= \sum_{s \in \text{Chi}(u)} \lambda_s w^{|s|} = \left(\sum_{s \in \text{Chi}(u)} \lambda_s \right) w^{|u|+1} = w w^{|u|} = w \mathbf{V}_w(e_u), \quad u \in V. \end{aligned}$$

Hence, applying linearity, we see that the equality in (13) holds with every $f \in \mathcal{E}$. This, continuity of \mathbf{V}_w and density of \mathcal{E} in $\ell^2(\beta)$ implies (13). \square

Proposition 16. Assume $(\star\star)$. Let S_λ be a weighted shift on \mathcal{T}_β . Then $\text{bpe}(\mathcal{T}_\beta) \subseteq \sigma_p(S_\lambda^*)$.

Proof. Suppose $w \in \text{bpe}(\mathcal{T}_\beta)$. Then, by Theorem 12, $k_w \in \ell^2(\beta)$. According to (12) and (13), we have

$$\begin{aligned} \langle f, S_\lambda^* k_w \rangle_\beta &= \langle S_\lambda f, k_w \rangle_\beta = \mathbf{V}_w(S_\lambda f) \\ &= w \mathbf{V}_w(f) = w \langle f, k_w \rangle_\beta = \langle f, \overline{w} k_w \rangle_\beta, \quad f \in \ell^2(\beta). \end{aligned}$$

Hence $S_\lambda^* k_w = \overline{w} k_w$ and as a consequence $\overline{w} \in \sigma_p(S_\lambda^*)$. Applying Corollary 13 we get the claim. \square

The following example shows that, in contrast to classical weighted shift, the inclusion in theorem above may be proper (see [16, Theorem 10 (i)]).

Example 17. Let $\mathcal{T}_{2,0} = (V_{2,0}, E_{2,0})$ be the directed tree defined by (see Figure 1)

$$V_{2,0} = \{(0,0)\} \cup \{(i,j) : i \in \{1,2\}, j \in \mathbb{N}\},$$

$$E_{2,0} = \left\{ ((0,0), (i,1)) : i \in \{1,2\} \right\} \cup \left\{ ((i,j), (i,j+1)) : i \in \{1,2\}, j \in \mathbb{N} \right\}.$$

Clearly, $\mathcal{T}_{2,0}$ is leafless and rooted. Let $\lambda = \{\lambda_v\}_{v \in V_{2,0}^\circ}$ and $\beta = \{\beta_v\}_{v \in V_{2,0}}$ be given by

$$\lambda_{(i,j)} = \begin{cases} \frac{1}{2} & \text{if } i \in \{1,2\} \text{ and } j = 1, \\ 1 & \text{if } i \in \{1,2\} \text{ and } j > 1, \end{cases}$$

and

$$\beta_{(i,j)} = \begin{cases} 1 & \text{if } i = 1 \text{ and } j \geq 1 \text{ or } i = j = 0, \\ \frac{1}{4^j} & \text{if } i = 2 \text{ and } j \geq 1. \end{cases}$$

Then $\mathcal{T}_{2,0}$, β and λ fulfill condition $(\star\star)$. By Theorem 12 we get $\text{bpe}(\mathcal{T}_{2,0}, \beta) = \{z : |z| < \frac{1}{2}\}$. On the other hand, by Proposition 28 below, the weighted shift S_λ on $(\mathcal{T}_{2,0}, \beta)$ is unitarily equivalent to the weighted shift S_μ on $(\mathcal{T}_{2,0}, \mathbb{1})$, where $\mu = \{\mu_v\}_{v \in V_{2,0}^\circ}$ is given by

$$\mu_{(i,j)} = \begin{cases} \frac{1}{2} & \text{if } i = 1 \text{ and } j = 1, \\ 1 & \text{if } i = 1 \text{ and } j > 1, \\ \frac{1}{4} & \text{if } i = 2 \text{ and } j = 1, \\ \frac{1}{2} & \text{if } i = 2 \text{ and } j > 1. \end{cases}$$

For $\theta \in \{z \in \mathbb{C} : |z| < 1\}$, let $k_\theta : V_{2,0} \rightarrow \mathbb{C}$ be defined by

$$k_\theta(i, j) = \begin{cases} \frac{1}{2} & \text{if } i = 0 \text{ and } j = 0, \\ \theta^j & \text{if } i = 1 \text{ and } j > 0, \\ 0 & \text{if } i = 2 \text{ and } j > 0. \end{cases}$$

Then k_θ is an eigenvector of S_μ corresponding to an eigenvalue θ . Hence $\{z : |z| < 1\} \subseteq \sigma_p(S_\mu^*) = \sigma_p(S_\lambda^*)$. In fact we can show that $\{z : |z| < 1\} = \sigma_p(S_\mu^*)$. Indeed, suppose that $\theta \in \{z \in \mathbb{C} : |z| \geq 1\}$ is an eigenvalue of S_μ^* corresponding to an eigenvector $k \in \ell^2(V, \mathbb{1})$. Then $(S_\mu^*)^n k = \theta^n k$ for every $n \in \mathbb{N}$, which implies that for every $(i, j) \in V_{2,0}$, $k(i, j) = \alpha_i \theta^j$ with some $\alpha_i \in \mathbb{C}$. This and $k \in \ell^2(V_{2,0}, \mathbb{1})$ yield $k = 0$, a contradiction.

According to Proposition 14 and Proposition 16 we see that $\text{bpe}(\mathcal{T}_\beta) \subseteq \Delta_{r(S_\lambda)}$ whenever S_λ is normaloid (i.e., $r(S_\lambda) = \|S_\lambda\|$). If S_λ is non-normaloid, then the following question arises.

Problem 18. *Does there exists $\hat{\varphi} \in \mathcal{M}(\mathcal{T}_\beta, \lambda)$ such that $\text{bpe}(\mathcal{T}_\beta) = \overline{\Delta_{r(S_\lambda)}}$ and $\sum_{k=0}^\infty \hat{\varphi}(k) w^k$ is divergent for any $w \in \partial \text{bpe}(\mathcal{T}_\beta)$?*

In view of Propositions 6 and 16 (cf. Problem 18), for any $\hat{\varphi} \in \mathcal{M}(\mathcal{T}_\beta, \lambda)$ we may define

$$\varphi(w) = \sum_{n \in \mathbb{N}_0} \hat{\varphi}(n) w^n, \quad w \in \text{int}(\text{bpe}(\mathcal{T}_\beta)).$$

Clearly, φ is analytic.

Theorem 19. *Let $\mathcal{T} = (V, E)$ be a countably infinite rooted and leafless directed tree, $\beta = \{\beta_v\}_{v \in V} \subseteq (0, \infty)$ and $\lambda = \{\lambda_v\}_{v \in V^\circ} \subseteq (0, \infty)$. If $\sum_{v \in \text{Chi}(u)} \lambda_v = 1$ for every $u \in V$ and $S_\lambda \in \mathbf{B}(\ell^2(\beta))$, then*

$$(14) \quad \mathbf{V}_w(M_{\hat{\varphi}} f) = \varphi(w) \mathbf{V}_w(f), \quad f \in \ell^2(\beta), \quad \hat{\varphi} \in \mathcal{M}(\mathcal{T}_\beta, \lambda), \quad w \in \text{int}(\text{bpe}(\mathcal{T}_\beta)).$$

Proof. Fix $w \in \text{int}(\text{bpe}(\mathcal{T}_\beta))$, $\hat{\varphi} \in \mathcal{M}(\mathcal{T}_\beta, \lambda)$ and $f \in \ell^2(\beta)$. For $n \in \mathbb{N}$, let $\hat{\varphi}^{(n)} : \mathbb{N}_0 \rightarrow \mathbb{C}$ be given by $\hat{\varphi}^{(n)}(k) = \chi_{\{1, \dots, n\}}(k) \hat{\varphi}(k)$. Since $S_\lambda \in \mathbf{B}(\ell^2(\beta))$, $\hat{\varphi}^{(n)} \in \mathcal{M}(\mathcal{T}_\beta, \lambda)$ by Theorem 5 (ii).

By induction and linearity from (13) we get

$$(15) \quad \mathbf{V}_w(M_{\hat{\varphi}^{(n)}} f) = \varphi^{(n)}(w) \mathbf{V}_w(f).$$

Since $w \in \text{int}(\text{bpe}(\mathcal{T}_\beta))$, the series $\sum_{v \in V} f(v) w^{|v|}$ is absolutely summable by Theorem 12, whence the series $\sum_{n \in \mathbb{N}_0} \hat{\varphi}^{(n)} w^n$ is absolutely convergent by Proposition 6. Therefore we can write

$$\begin{aligned} \varphi(w) \mathbf{V}_w(f) &= \left(\sum_{k \in \mathbb{N}_0} \hat{\varphi}(k) w^k \right) \left(\sum_{v \in V} f(v) w^{|v|} \right) \\ &= \left(\sum_{k \in \mathbb{N}_0} \hat{\varphi}(k) w^k \right) \left(\sum_{l \in \mathbb{N}_0} \sum_{v \in \text{Chi}^{(l)}(\text{root})} f(v) w^l \right) = \sum_{m \in \mathbb{N}_0} \alpha_{\hat{\varphi}}(m) w^m, \end{aligned}$$

with $\alpha_{\hat{\varphi}}: \mathbb{N}_0 \rightarrow \mathbb{C}$ given by the Cauchy product formula. In the same manner, we see that

$$\varphi^{(n)}(w) \mathbf{V}_w(f) = \left(\sum_{k \in \mathbb{N}_0} \hat{\varphi}^{(n)}(k) w^k \right) \left(\sum_{l \in \mathbb{N}_0} \sum_{v \in \text{Chi}^{(l)}(\text{root})} f(v) w^l \right) = \sum_{m \in \mathbb{N}_0} \alpha_{\hat{\varphi}^{(n)}}(m) w^m,$$

with $\alpha_{\hat{\varphi}^{(n)}}: \mathbb{N}_0 \rightarrow \mathbb{C}$. It is easily seen that

$$(16) \quad \alpha_{\hat{\varphi}^{(n)}}(m) = \alpha_{\hat{\varphi}}(m), \quad m \leq n, \quad n \in \mathbb{N}_0.$$

By Theorem 12, both the series $\sum_{v \in V} (M_{\hat{\varphi}} f(v)) w^{|v|} = V_w(M_{\hat{\varphi}} f)$ and $\sum_{v \in V} (M_{\hat{\varphi}^{(n)}} f(v)) w^{|v|} = V_w(M_{\hat{\varphi}^{(n)}} f)$ are absolutely summable. Moreover, for every $n \in \mathbb{N}_0$ and $v \in V$ such that $|v| \leq n$ we have

$$(M_{\hat{\varphi}} f)(v) = \sum_{k=0}^{|v|} \lambda_{\text{par}^k(v)|v} \hat{\varphi}(k) f(\text{par}^k(v)) = \sum_{k=0}^{|v|} \lambda_{\text{par}^k(v)|v} \hat{\varphi}^{(n)}(k) f(\text{par}^k(v)) = (M_{\hat{\varphi}^{(n)}} f)(v),$$

which implies that coefficients of the series

$$\sum_{v \in V} (M_{\hat{\varphi}} f(v)) w^{|v|} \quad \text{and} \quad \sum_{v \in V} (M_{\hat{\varphi}^{(n)}} f(v)) w^{|v|}$$

at $v \in V$ with $|v| \leq n$ are the same. Combining this, (16) and (15) gives (14). \square

Proposition 20. *Assume $(\star\star)$. For every $\hat{\varphi} \in \mathcal{M}(\mathcal{T}_\beta, \lambda)$, $\varphi(\text{int}(\text{bpe}(\mathcal{T}_\beta)))^* \subseteq \sigma_p(M_{\hat{\varphi}}^*)$, where $\Omega^* = \{\bar{z}: z \in \Omega\}$.*

Proof. Fix $w \in \text{int}(\text{bpe}(\mathcal{T}_\beta))$. Then, by Theorem 12, $k_w \in \ell^2(\beta)$. Thus, applying Theorem 12 and (14), we have

$$\begin{aligned} \langle f, M_{\hat{\varphi}}^* k_w \rangle_\beta &= \langle M_{\hat{\varphi}} f, k_w \rangle_\beta = \mathbf{V}_w(M_{\hat{\varphi}} f) \\ &= \varphi(w) \mathbf{V}_w(f) = \varphi(w) \langle f, k_w \rangle_\beta = \langle f, \overline{\varphi(w)} k_w \rangle_\beta, \quad f \in \ell^2(\beta). \end{aligned}$$

Hence $M_{\hat{\varphi}}^* k_w = \overline{\varphi(w)} k_w$, and as a consequence $\overline{\varphi(w)} \in \sigma_p(M_{\hat{\varphi}}^*)$. \square

Remark 21. As regards Proposition 20, it is worth noting that if \mathcal{T} is isomorphic to the directed tree $(\mathbb{N}_0, \{(n, n+1): n \in \mathbb{N}_0\})$, then $\varphi(\text{bpe}(\mathcal{T}_\beta))^* \subseteq \sigma_p(M_{\hat{\varphi}}^*)$ (see [16, Theorem 10]).

6. POINT SPECTRUM OF S_λ^* VIA PATHS

In this section we aim to show that some information about point spectrum of S_λ^* can be deduced from the behaviour of S_λ^* on paths.

We begin with easy observation that the point spectrum of S_λ^* contains all the bounded point evaluations calculated with respect to paths, defined in the following natural way: if $\mathcal{T} = (V, E)$ is a countably directed tree, $\mathcal{P} = (V_{\mathcal{P}}, E_{\mathcal{P}})$ is a path in \mathcal{T} and $\beta = \{\beta_v\}_{v \in V} \subseteq (0, \infty)$, then

$\text{bpe}(\mathcal{P}_\beta)$, the set of *bounded point evaluations with respect to \mathcal{P}* , is defined as the set of all $w \in \mathbb{C}$ such that $\sum_{v \in V_\mathcal{P}} \beta_v^{-1} |w|^{2|v|} < \infty$.

Proposition 22. *Assume $(\star\star)$. Let S_λ be a weighted shift on \mathcal{T}_β . Then the following assertions hold:*

- (i) *for every $\mathcal{P} \in \mathcal{P}$, $\text{bpe}(\mathcal{P}_\beta) \subseteq \sigma_p(\mathcal{Q}_\mathcal{P} S_\lambda^*|_{\ell^2(\mathcal{P}, \beta)})$,*
- (ii) *$\text{bpe}(\mathcal{T}_\beta) \subseteq \bigcup_{\mathcal{P} \in \mathcal{P}} \text{bpe}(\mathcal{P}_\beta) \subseteq \sigma_p(S_\lambda^*)$,*
- (iii) *if $\text{card}(\mathcal{P}(\mathcal{T})) < \infty$, then $\text{bpe}(\mathcal{T}_\beta) = \bigcup_{\mathcal{P} \in \mathcal{P}} \text{bpe}(\mathcal{P}_\beta)$.*

Proof. Arguing as in the proof of Proposition 16 we show that (i). The first inclusion in assertion (ii) and assertion (iii) follow immediately from the definitions of $\text{bpe}(\mathcal{T}_\beta)$ and $\text{bpe}(\mathcal{P}_\beta)$, $\mathcal{P} \in \mathcal{P}$. Then, the rest follows easily from the fact that for every $\mathcal{P} = (V_\mathcal{P}, E_\mathcal{P}) \in \mathcal{P}$, $\sigma_p(\mathcal{Q}_\mathcal{P} S_\lambda^*|_{\ell^2(\mathcal{P}, \beta)}) \subseteq \sigma_p(S_\lambda^*)$ (see Lemma 2). \square

Another and more concrete way of localizing the point spectrum of S_λ is available. One definition more is required: assuming (\star) , we set

$$r_2^\mathcal{P}(S_\lambda) = \lim_{n \rightarrow \infty} \inf \left\{ \left(\sqrt{\beta_v} \lambda_{\text{root}|v}^\mathcal{P} \right)^{\frac{1}{|v|}} : v \in \mathcal{P}, |v| \geq n \right\}, \quad \mathcal{P} \in \mathcal{P},$$

$$r_2^+(S_\lambda) = \sup \left\{ r_2^\mathcal{P}(S_\lambda) : \mathcal{P} \in \mathcal{P} \right\}.$$

Theorem 23. *Let $\mathcal{T} = (V, E)$ be a countably infinite rooted and leafless directed tree, $\beta = \{\beta_v\}_{v \in V} \subseteq (0, \infty)$ and $\{\lambda_v\}_{v \in V^\circ} \subseteq (0, \infty)$. Let $S_\lambda \in \mathbf{B}(\ell^2(\beta))$ be a weighted shift on \mathcal{T}_β . Then $\Delta_{r_2^+(S_\lambda)} \subseteq \sigma_p(S_\lambda^*)$.*

Proof. Fix $\mathcal{P} \in \mathcal{P}$. Then for every $k \in \mathbb{N}_0$ there exists unique $v_k \in \mathcal{P}$ such that $|v_k| = k$. Define the sequence $\mu = \{\mu_k\}_{k=0}^\infty \subseteq (0, \infty)$ by $\mu_k = \sqrt{\frac{\beta_{v_{k+1}}}{\beta_{v_k}}} \lambda_{v_{k+1}}$. Let S_μ be the classical weighted shift on $\ell^2(\mathbb{N}_0)$ with weights μ . Then one can easily show using Lemma 2 that S_μ^* is unitarily equivalent to $\mathcal{Q}_\mathcal{P} S_\lambda^*|_{\ell^2(\mathcal{P}, \beta)}$ via $U: \ell^2(\mathbb{N}_0) \rightarrow \ell^2(\mathcal{P}, \beta)$ given by $(Uf)(u) = \beta_u^{-1/2} f(u)$ for $u \in V$, $f \in \ell^2(\mathbb{N}_0)$. According to [16, Theorem 8], we have

$$(17) \quad \Delta_{r_2(\mathcal{P})} \subseteq \sigma_p(S_\mu^*) \subseteq \overline{\Delta_{r_2(\mathcal{P})}},$$

with $r_2(\mathcal{P}) = \liminf_{k \rightarrow \infty} (\mu_0 \cdots \mu_k)^{\frac{1}{k+1}}$. Clearly, $r_2^\mathcal{P}(S_\lambda) = r_2(\mathcal{P})$. This, (17) and Lemma 2 imply that

$$\Delta_{r_2^+(S_\lambda)} \subseteq \bigcup_{\mathcal{P} \in \mathcal{P}} \sigma_p(\mathcal{Q}_\mathcal{P} S_\lambda^*|_{\ell^2(\mathcal{P}, \beta)}) \subseteq \sigma_p(S_\lambda^*),$$

which completes the proof. \square

Using the unitary equivalence of operators $\mathcal{Q}_\mathcal{P} S_\lambda^*|_{\ell^2(\mathcal{P}, \beta)}$ and S_μ^* and applying [16, Proposition 7 and Theorem 10 (i)] we get

Corollary 24. *Assume (\star) . Let $S_\lambda \in \mathbf{B}(\ell^2(\beta))$ be a weighted shift on \mathcal{T}_β . Then $\Delta_{r_2^\mathcal{P}(S_\lambda)} = \sigma_p(\mathcal{Q}_\mathcal{P} S_\lambda^*|_{\ell^2(\mathcal{P}, \beta)})$ for all \mathcal{P} .*

In view of [16, Theorem 8], it is reasonable to ask for a solution of the following.

Problem 25. *When does the inclusion*

$$\sigma_p(S_\lambda^*) \subseteq \overline{\Delta_{r_2^+(S_\lambda)}}$$

hold?

As shown below the inclusion of Problem 25 is satisfied if the tree has finitely many branching vertexes. Recall that a vertex v is called a *branching vertex* if the set $\text{Chi}(v)$ contains at least two distinct vertices.

Proposition 26. *Assume $(\star\star)$. If the directed tree \mathcal{T} has finitely many branching points, then $\sigma_p(S_\lambda^*) \subseteq \overline{\Delta_{r_2^+}(S_\lambda)}$.*

Proof. Since \mathcal{T} has finite number of branching vertices, there is $n_0 \in \mathbb{N}$ such that $\text{Chi}(\text{par}(v)) = \{v\}$ for every $v \in V$ with $|v| \geq n_0$. Let $w \in \sigma_p(S_\lambda^*)$ and let $f \in \ell^2(\beta)$ be the corresponding eigenvector, i.e.,

$$(18) \quad wf(u) = \sum_{v \in \text{Chi}(u)} \lambda_v \frac{\beta_v}{\beta_u} f(v), \quad u \in V.$$

We claim that for every $\mathcal{P} \in \mathcal{P}$ there is $\alpha_{\mathcal{P}} \in \mathbb{C}$ such that

$$(19) \quad f(v) = \alpha_{\mathcal{P}} \frac{w^{|v|}}{\beta_v}, \quad v \in \{u \in \mathcal{P} : |u| \geq n_0\}.$$

Indeed, fix $\mathcal{P} \in \mathcal{P}$. Let $v_{\mathcal{P}}$ be the only vertex in \mathcal{P} such that $|v_{\mathcal{P}}| = n_0$ and let v_k , $k \in \mathbb{N}$, be the unique vertex in \mathcal{P} such that $|v_k| = n_0 + k$. In view of $(\star\star)$, $\lambda_{v_k} = 1$ for every $k \in \mathbb{N}$. Then, applying (18) repeatedly k -times, we deduce that

$$f(v_k) = w^k \frac{\beta_{v_{\mathcal{P}}}}{\beta_{v_k}} f(v_{\mathcal{P}}), \quad k \in \mathbb{N},$$

which clearly gives (19) with $\alpha_{\mathcal{P}} = w^{-|v_{\mathcal{P}}|} \beta_{v_{\mathcal{P}}} f(v_{\mathcal{P}})$.

Since f is non-zero, there is $\tilde{\mathcal{P}} \in \mathcal{P}$ such that $f(v_{\tilde{\mathcal{P}}}) \neq 0$. In view of (19), the inequalities

$$\alpha_{\tilde{\mathcal{P}}}^2 \sum_{k=1}^{\infty} \left(\frac{w^k}{\beta_{v_k}} \right)^2 \beta_{v_k} \leq \sum_{v \in V} |f(v)|^2 \beta_v < \infty,$$

Theorem 12 and Proposition 22 we have $w \in \text{bpe}(\mathcal{P}_{\beta}) \subseteq \sigma_p(\mathcal{Q}_{\mathcal{P}} S_\lambda^*|_{\ell^2(\tilde{\mathcal{P}}, \beta)}) = \sigma_p(S_\mu^*)$, where S_μ is as in the proof of Theorem 23. Hence, by (17) we have $|w| \leq r_2^+(\tilde{\mathcal{P}})(S_\lambda) \leq r_2^+(S_\lambda)$. This proves the claim. \square

On the other hand, if there is plenty of branching vertexes in each generation, then the inclusion of Problem 25 does not hold. This follows from the Proposition 27 below. Recall that for $\kappa \in \mathbb{N}$, the rooted κ -ary directed tree is the directed tree $\mathcal{T}^{(\kappa)} = (V^{(\kappa)}, E^{(\kappa)})$ given by

$$V^{(\kappa)} = \{(k, l) : k \in \mathbb{N}_0 \text{ and } l \in J_{\kappa^k}\},$$

$$E^{(\kappa)} = \left\{ ((k, l), (k+1, m)) : k \in \mathbb{N}_0, l \in J_{\kappa^k}, m = \kappa(l-1) + 1, \dots, \kappa l \right\}.$$

Figure 2 above gives a more intuitive description of κ -ary directed trees.

Proposition 27. *Let $\kappa \in \mathbb{N}$. Let S_λ be the weighted shift on $\mathcal{T}_\beta^{(\kappa)}$ with $\lambda_v = \kappa^{-1}$ for $v \in V^{(\kappa)} \setminus \{(0, 1)\}$ and $\beta_v = \kappa^{-|v|}$ for all $v \in V^{(\kappa)}$. Then $S_\lambda \in \mathbf{B}(\ell^2(\beta))$, $r_2^+(S_\lambda) = \frac{1}{\sqrt{\kappa^3}}$ and $\{z \in \mathbb{C} : |z| < \frac{1}{\kappa}\} \subseteq \sigma_p(S_\lambda^*)$.*

Proof. Direct calculation shows that $S_\lambda \in \mathbf{B}(\ell^2(\beta))$ and $r_2^+(S_\lambda) = \frac{1}{\sqrt{\kappa^3}}$. Now, for $w \in \mathbb{C}$, let us define the function $f_w : V \rightarrow \mathbb{C}$ by $f_w(v) = w^{|v|}$ for $v \in V^{(\kappa)}$. Clearly,

$$\|f_w\|_\beta^2 = \sum_{v \in V^{(\kappa)}} \frac{|w|^{2|v|}}{\kappa^{|v|}} = \sum_{n=0}^{\infty} |w|^{2n} < \infty \quad \text{for } |w| < 1,$$

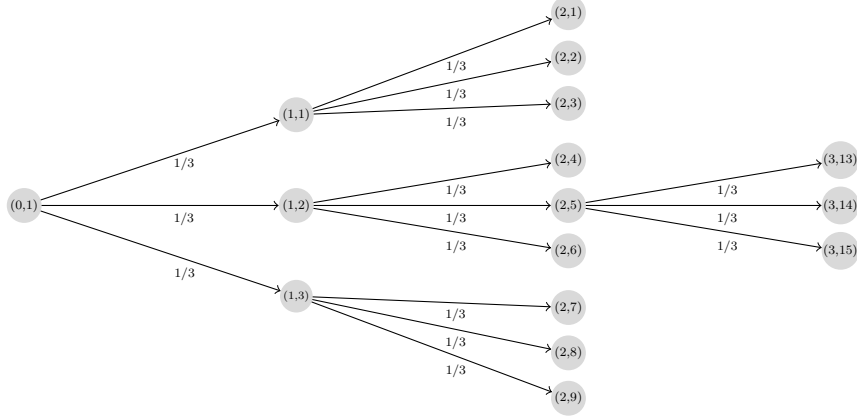


FIGURE 2. 3-ary directed tree

and thus (see Lemma 2)

$$(S_{\lambda}^* f_w)(u) = \sum_{v \in \text{Chi}(u)} \frac{1}{\kappa^2} f_w(v) = \frac{w^{|u|+1}}{\kappa} = \frac{w}{\kappa} f_w(u) \quad \text{for } u \in V^{(\kappa)} \text{ and } |w| < 1.$$

This proves the required inclusion. \square

7. NON-WEIGHTED DIRECTED TREES

In this section we concentrate on weighted shifts on directed trees \mathcal{T}_1 and comment on how the results of the previous sections apply in this context.

We begin by showing that a weighted shift S_{μ} on \mathcal{T}_1 is unitarily equivalent to a weighted shift S_{λ} on \mathcal{T}_{β} with any given weights λ . The proof is essentially extracted from [8, Example 4.3.1].

Proposition 28. *Let $\mathcal{T} = (V, E)$ be a countably infinite rooted and leafless directed tree and let $\mu = \{\mu_v\}_{v \in V^\circ}$ be a family of nonzero complex numbers. Let $\lambda = \{\lambda_v\}_{v \in V^\circ} \subseteq (0, \infty)$. Define the family $\beta = \{\beta_v\}_{v \in V} \subseteq (0, \infty)$ by*

$$\beta_v = \left| \frac{\mu_{\text{root}|v}}{\lambda_{\text{root}|v}} \right|^2, \quad v \in V.$$

Then the operator $U: \ell^2(V, \mathbb{1}) \rightarrow \ell^2(V, \beta)$, defined by

$$(Uf)(u) = \frac{\lambda_{\text{root}|u}}{\mu_{\text{root}|u}} f(u), \quad u \in V, \quad f \in \ell^2(\mathbb{1}).$$

is unitary and the weighted shift S_{μ} on \mathcal{T}_1 is unitarily equivalent to the weighted shift S_{λ} on \mathcal{T}_{β} via U , i.e.,

$$US_{\mu} = S_{\lambda}U.$$

Proof. Since, for every $f \in \ell^2(\mathbb{1})$ we have

$$\sum_{u \in V} |(Uf)(u)|^2 \beta_u = \sum_{u \in V} \left| \frac{\lambda_{\text{root}|u}}{\mu_{\text{root}|u}} \right|^2 |f(u)|^2 \beta_u = \sum_{u \in V} |f(u)|^2,$$

we see that U is well-defined unitary isomorphism.

Let $f \in \ell^2(\mathbb{1})$ be such that $\Lambda_{\mathcal{T}}^{\mu} f \in \ell^2(\mathbb{1})$. Thus we get

$$(20) \quad (U\Lambda_{\mathcal{T}}^{\mu} f)(u) = \frac{\lambda_{\text{root}|u}}{\mu_{\text{root}|u}} (\Lambda_{\mathcal{T}}^{\mu} f)(u) = \frac{\lambda_{\text{root}|u}}{\mu_{\text{root}|u}} \mu_u f(\text{par}(u)) \\ = \lambda_u \frac{\lambda_{\text{root}|\text{par}(u)}}{\mu_{\text{root}|\text{par}(u)}} f(\text{par}(u)) = \lambda_u (Uf)(\text{par}(u)) = (\Lambda_{\mathcal{T}}^{\lambda} Uf)(u), \quad u \in V \setminus \{\text{root}\}.$$

On the other hand, $(U\Lambda_{\mathcal{T}}^{\mu} f)(\text{root}) = 0$. This and (20) imply that $Uf \in \mathcal{D}(S_{\lambda})$. Moreover, $US_{\mu}f = S_{\lambda}Uf$. Therefore, we obtain the inclusion $US_{\mu} \subseteq S_{\lambda}U$.

Now, suppose that $f \in \ell^2(\mathbb{1})$ satisfy $Uf \in \mathcal{D}(S_{\lambda})$. Set $g(u) = \frac{\lambda_{\text{root}|u}}{\mu_{\text{root}|u}} (\Lambda_{\mathcal{T}}^{\mu} f)(u)$ for $u \in V$. Then, by (20), $g \in \ell^2(V, \beta)$ and consequently $(U^{-1}g)(u) = (\Lambda_{\mathcal{T}}^{\mu} f)(u)$ for every $u \in V$. This means that $f \in \mathcal{D}(S_{\mu})$. Combining this with the inclusion $US_{\mu} \subseteq S_{\lambda}U$ we complete the proof. \square

In Sections 5 and 6 we considered weighted shifts with weights summing over children of every vertex up to 1 (see condition $(\star\star)$). In view of the following Proposition we didn't lose much of generality doing so.

Proposition 29. *Let $\mathcal{T} = (V, E)$ be a countably infinite rooted and leafless directed tree and $\mu = \{\mu_v\}_{v \in V^{\circ}} \subseteq \mathbb{C} \setminus \{0\}$. Suppose*

$$\mu_{[v]} := \sum_{u \in \text{Chi}(\text{par}(v))} |\mu_u|^2 < \infty, \quad v \in V^{\circ}.$$

Define the families $[\mu] := \{[\mu]_v\}_{v \in V^{\circ}}$ and $\langle \mu \rangle := \{\langle \mu \rangle_v\}_{v \in V}$ by

$$[\mu]_v = \frac{|\mu_v|^2}{\mu_{[v]}}, \quad \langle \mu \rangle_v = \left| \frac{\mu_{[\text{root}|v]}}{\mu_{\text{root}|v}} \right|^2 \quad \text{with} \quad \mu_{[u|v]} = \begin{cases} 1 & \text{if } u = v, \\ \prod_{j=0}^{n-1} \mu_{[\text{par}^j(v)]} & \text{if } v \in \text{Chi}^{(n)}(u). \end{cases}$$

Then the weighted shift $S_{[\mu]}$ on $\mathcal{T}_{\langle \mu \rangle}$ is unitarily equivalent to the weighted shift S_{μ} on $\mathcal{T}_{\mathbb{1}}$. Moreover, for every $u \in V$, $e_u \in \mathcal{D}(S_{[\mu]})$ and $S_{[\mu]}e_u = \sum_{v \in \text{Chi}(u)} \frac{|\mu_v|^2}{\mu_{[v]}} e_v$.

Proof. Apply Lemma 28 and [7, Proposition 3.1.3]. \square

In view of the above, when investigating weighted shifts S_{μ} on $\mathcal{T}_{\mathbb{1}}$ we can always pass on to $S_{[\mu]}$ on $\mathcal{T}_{\langle \mu \rangle}$ whenever a property we are interested in is invariant under unitary equivalence. Note that, assuming S_{μ} is bounded, the families $\lambda := [\mu]$ and $\beta := \langle \mu \rangle$ satisfy $(\star\star)$. This enables us to apply most of the results of previous sections.

Remark 30. As regards Proposition 29 it is worth also to notice that any weighted shift S_{λ} on a weighted directed tree \mathcal{T}_{β} is unitarily equivalent to a weighted shift S_{μ} on a directed tree $\mathcal{T}_{\mathbb{1}}$. The equivalence is given by the operator $W: \ell^2(\beta) \rightarrow \ell^2(\mathbb{1})$, defined by

$$(Wf)(u) = \sqrt{\beta_u} f(u), \quad u \in V, \quad f \in \ell^2(\beta),$$

whence the formula for weights μ is

$$\mu_v = \sqrt{\frac{\beta_v}{\beta_{\text{par}(v)}}} \lambda_v, \quad v \in V^{\circ}.$$

Now we observe that, keeping the notation from Proposition 28, the multiplication operators $M_{\hat{\varphi}}^{\lambda, \beta}$ and $M_{\hat{\varphi}}^{\mu, \mathbb{1}}$ related to S_{λ} and S_{μ} , respectively, are unitarily equivalent via the operator U .

Proposition 31. *Under the assumptions of Proposition 28 the following condition is satisfied*

$$(\Upsilon \Gamma_{\hat{\varphi}}^{\lambda} f)(v) = (\Gamma_{\hat{\varphi}}^{\mu} \Upsilon f)(v), \quad v \in V, \quad f \in \mathbb{C}^V,$$

where $\Upsilon: \mathbb{C}^V \rightarrow \mathbb{C}^V$ be given by

$$(\Upsilon f)(u) = \frac{\mu_{\text{root}|u}}{\lambda_{\text{root}|u}} f(u), \quad u \in V, \quad f \in \mathbb{C}^V.$$

Moreover, Υ induces operator $U^*: \ell^2(\beta) \rightarrow \ell^2(\mathbb{1})$ and

$$U^* M_{\hat{\varphi}}^{\lambda, \beta} = M_{\hat{\varphi}}^{\mu, \mathbb{1}} U^*.$$

By applying Propositions 28 and 31, we deduce that S_{λ} and S_{μ} induce the same multiplier algebra.

Corollary 32. *Under the assumptions of Proposition 28, the algebras $\mathcal{M}(\mu, \mathcal{T}_{\beta})$ and $\mathcal{M}(\lambda, \mathcal{T}_{\mathbb{1}})$ coincide.*

Remark 33. Under the assumptions of Proposition 28, the operators $M_{\hat{\varphi}}^{\lambda, \beta}$ and $M_{\hat{\varphi}}^{|\lambda|, \beta}$, where $|\lambda| = \{|\lambda_v|\}_{v \in V^\circ}$, are unitarily equivalent.

As a consequence of Corollary 32 and the unitary equivalence between S_{λ} and S_{μ} one can easily derive counterparts of all the results of Section 4 for S_{μ} and $\mathcal{M}(\mu, \mathbb{1})$. In particular, in view of Proposition 29, we have

$$(21) \quad \mathcal{M}(\mu, \mathcal{T}_{\mathbb{1}}) = \mathcal{M}([\mu], \mathcal{T}_{\langle \mu \rangle}).$$

As regards Section 5 we see that replacing β by $\langle \mu \rangle$ in Theorem 12 we get

$$\text{bpe}(\mathcal{T}_{\langle \mu \rangle}) = \left\{ w \in \mathbb{C} : \sum_{v \in V} \left| \frac{\mu_{\text{root}|v}}{\mu_{[\text{root}|v]}} \right|^2 |w|^{2|v|} < \infty \right\}.$$

Combining this with Proposition 31 and Corollary 32 (with $[\mu]$ and $\langle \mu \rangle$ in place of λ and β , respectively) we deduce the following

Corollary 34. *Under the assumptions of Proposition 28, if $S_{\mu} \in \mathbf{B}(\ell^2(\mathbb{1}))$, then*

$$\tilde{V}_w(M_{\hat{\varphi}}^{\mu, \mathbb{1}} f) = \varphi(w) \tilde{V}_w(f), \quad f \in \ell^2(\mathbb{1}), \quad \hat{\varphi} \in \mathcal{M}(\mu, \mathcal{T}_{\mathbb{1}}), \quad w \in \text{int}(\text{bpe}(\mathcal{T}_{\langle \mu \rangle})),$$

where $\tilde{V}_w: \ell^2(\mathbb{1}) \rightarrow \mathbb{C}$ is given by $\tilde{V}_w = V_w \circ U$.

Using (21) and Proposition 20 we get the following.

Corollary 35. *Under the assumptions of Proposition 28, if $S_{\mu} \in \mathbf{B}(\ell^2(\mathbb{1}))$, then*

$$\varphi\left(\text{int}(\text{bpe}(\mathcal{T}_{\langle \mu \rangle}))\right)^* \subseteq \sigma_p\left(\left(M_{\hat{\varphi}}^{\mu, \mathbb{1}}\right)^*\right), \quad \hat{\varphi} \in \mathcal{M}(\mu, \mathcal{T}_{\mathbb{1}}).$$

Regarding Section 6 we note that all the results apply to the weighted shift S_{μ} on $\mathcal{T}_{\mathbb{1}}$.

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The models [4], obtained independently, rely on the work [17] and enable, under additional assumptions, representing a weighted shift on a directed tree as a multiplication operator M_z on

a reproducing kernel Hilbert space \mathcal{H} of E -valued holomorphic functions on a disc centered at the origin, where $E = \ker S_\lambda^*$.

REFERENCES

- [1] P. Budzyński, P. Dymek, Z. J. Jabłoński, J. Stochel, *Subnormal weighted shifts on directed trees and composition operators in L^2 -spaces with non-densely defined powers*, Abs. Appl. Anal. **2014** (2014), Article ID 791817, 6 pages.
- [2] P. Budzyński, Z. J. Jabłoński, I. B. Jung, J. Stochel *Subnormal weighted shifts on directed trees whose n th powers have trivial domain*, J. Math. Anal. Appl. (to appear), doi:10.1016/j.jmaa.2015.10.021.
- [3] Conway, John B, A course in functional analysis, Graduate Texts in Mathematics, 96, Springer-Verlag, New York, 1990.
- [4] S. Chavan, S. Trivedi, *An analytic model for left-invertible weighted shifts on directed trees*, <http://arxiv.org/abs/1510.03075>.
- [5] R. Gellar, *Operators commuting with a weighted shift*, Proc. Amer. Math. Soc. **23** (1969), 538-545.
- [6] K. Hoffman, Banach Spaces of Analytic Functions, Prentice-Hall Series in Modern Analysis Prentice-Hall, Inc., Englewood Cliffs, N. J. 1962.
- [7] Z. J. Jabłoński, I. B. Jung, J. Stochel, *Weighted shifts on directed trees*, Mem. Amer. Math. Soc. **216** (2012), no. 1017, viii+107pp.
- [8] Z. J. Jabłoński, I. B. Jung, J. Stochel, *A non-hyponormal operator generating Stieltjes moment sequences*, J. Funct. Anal. **262** (2012), 3946-3980.
- [9] Z. J. Jabłoński, I. B. Jung, J. Stochel *Operators with absolute continuity properties: an application to quasnormality*, Stud. Math., **215** (2013), 11-30.
- [10] Z. J. Jabłoński, I. B. Jung, J. Stochel, *A hyponormal weighted shift on a directed tree whose square has trivial domain*, Proc. Amer. Math. Soc. **142** (2014), 3109-3116.
- [11] Z. J. Jabłoński, I. B. Jung, J. Stochel *Unbounded Quasinormal Operators Revisited*, Integr. Equ. Oper. Theory **79** (2014), 135-149.
- [12] N. P. Jewell, A. R. Lubin, *Commuting weighted shifts and analytic function theory in several variables*, J. Oper. Theory **1** (1979), 207-223.
- [13] I. B. Jung, J. Stochel, *Subnormal operators whose adjoints have rich point spectrum*, J. Funct. Anal. **255** (2008), 1797-1816.
- [14] N. K. Nikolskii, Treatise on the shift operator. Spectral function theory. With an appendix by S. V. Khrushchev and V. V. Peller. Translated from the Russian by Jaak Peetre. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 273. Springer-Verlag, Berlin, 1986.
- [15] P. Pietrzycki, *The single equality $A^{*n}A^n = (A^*A)^n$ does not imply the quasnormality of weighted shifts on rootless directed trees*, <http://arxiv.org/abs/1502.06396>.
- [16] A. L. Shields, *Weighted shift operators and analytic function theory*, Topics in operator theory, pp. 49-128. Math Surveys, No. 13, Amer. Math. Soc., Providence, R.I., 1974.
- [17] S. Shimorin, *Wold-type decompositions and wandering subspaces for operators close to isometries*, J. Reine Angew. Math. **531** (2001), 147-189
- [18] J. Stochel, F. H. Szafraniec, *On normal extensions of unbounded operators. III. Spectral properties*, Publ. RIMS, Kyoto Univ. **25** (1989), 105-139.
- [19] F. H. Szafraniec, *The reproducing kernel Hilbert space and its multiplication operators*, Operator Theory: Advances and Applications **114** (2000), 253-263
- [20] F. H. Szafraniec, *Multipliers in the reproducing kernel Hilbert space, subnormality and noncommutative complex analysis*, Operator Theory: Advances and Applications **143** (2003), 313-331.
- [21] J. Trepkowski, *Aluthge transforms of weighted shifts on directed trees*, J. Math. Anal. Appl. **425** (2015), 886-899.
- [22] D. Xia, *On the analytic model of a class of hyponormal operator*, Integral Eq. Oper. Theory **6** (1983), 134-157.

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